

# The Sobolev moment problem and Jordan dilations

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**ABSTRACT.** Moment problems and orthogonal polynomials, both meant in a single real variable, belong to the oldest problems in Classical Analysis. They have been developing for over a century in two parallel, mostly independent streams. During the last 20 years a rapid advancement of polynomials orthogonal in Sobolev space has been noticed, see [5] for an updated survey; their moment counterpart seems to be not paid enough attention as it deserves. In this paper we intend to resume the theme of [3] and also [4], and open the door for further, deeper study of the problem.

**1. Opening.** Solving a moment problem means, roughly speaking, to find a spectral representation of a given data. This is usually understood as a typical inverse spectral problem, also because of the traditional connection between moments and operators; the latter are the object of special concern in the present paper.

For example, given a sequence  $(a(n))_{n=0}^{\infty}$  of numbers, the classical (named after Hamburger) moment problem asks under which conditions there is a nonnegative Borel measure  $\mu$  on  $\mathbb{R}$  such that

$$a(n) = \int_{\mathbb{R}} t^n \mu(dt), \quad n = 0, 1, \dots \quad (1)$$

This happens if and only if

$$\sum_{k,l} a(k+l) \xi_m \bar{\xi}_n \geq 0, \quad \xi_n = 0 \text{ but a finite number of } n\text{'s}.$$

The above is referred to as positive definiteness of the matrix<sup>1</sup>  $(a(k+l))_{k,l=0}^{\infty}$ , which in turn can be viewed as positive definiteness of  $n \rightarrow a(n)$  considered as a function on the involution semigroup<sup>2</sup>  $\mathbb{N}$ . The important feature of the above is translational invariance

$$a(m+k, b) = a(m, n+k) \text{ for all } k \in \mathbb{N}$$

of the Hankel matrix  $(a(m, n))_{m,n=0}^{\infty}$ , here with the notation  $a(m, n) \stackrel{\text{def}}{=} \int_{\mathbb{R}} t^{m+n} \mu(dt)$ . This gives an opportunity to make use of harmonic analysis ideas on involution semigroups combined with operator dilation theory as successfully done in [18].

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<sup>1</sup> This is pretty often considered in terms of its determinants.

<sup>2</sup> We use  $\mathbb{N}$  for  $\{0, 1, \dots\}$ .

In analogy with the above, in the present paper we consider the following problem. Given a bisequence<sup>3</sup>  $(s(m, n))_{m, n=0}^{\infty}$  of numbers, we ask under which conditions there is a  $2 \times 2$  positive definite matrix  $\boldsymbol{\mu} \stackrel{\text{def}}{=} (\mu_{i,j})_{i,j=0}^1$  of measures on  $\mathbb{R}$  such that

$$s(m, n) = \sum_{i,j=0}^1 \int_{\mathbb{R}} t^{m(i)} t^{n(j)} \mu_{i,j}(dt), \quad m, n = 0, 1, \dots \quad (2)$$

The superscript  $(i)$  designates the  $i$ -th derivative and positive definiteness of the  $2 \times 2$  matrix valued function  $(\mu_{i,j}(\cdot))_{i,j=0}^1$  defined on Borel subsets of  $\mathbb{R}$  means that for every Borel  $\rho$  the  $2 \times 2$  complex matrix  $(\mu_{i,j}(\rho))_{i,j=0}^1$  is positive definite. Call this question the *Sobolev moment problem* as the integral on the right hand side of (2) is of Sobolev type. A bisequence  $s$  enjoying the property (2) will be called a *Sobolev moment sequence*. Let us remark that positive definiteness of the matrix  $\boldsymbol{\mu}$  does not exclude some of its entry measures  $\mu_{i,j}$  to be signed or complex, (cf. [9] for complex measures) though it forces the diagonal entries  $\mu_{i,i}$ ,  $i = 0, 1$  to be positive measures anyway.

The bisequence  $(s(m, n))_{m, n=0}^{\infty}$  is no longer translationally invariant (in Hankel sense) unlike in the case of the Hamburger moment problem. Another, not translationally invariant “moment problem” on the real line  $\mathbb{R}$  is treated<sup>4</sup> in [7]. There a dilation argument is exploited as well.

Our main tool in analysing the Sobolev moment problem will be the theory of positive definite forms, in the sense already defined and considered in [13]. Namely, we say that

$$\mathbf{t} : \mathbb{N} \times \mathbb{N} \times \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}$$

is a form over  $\mathbb{N} \times \mathbb{N} \times \mathbb{C}^2 \times \mathbb{C}^2$  (informally abbreviated below to  $\mathbb{N}^2 \times \mathbb{C}^4$ ) if for any  $m, n \in \mathbb{N}$  the mapping  $\mathbf{t}(m, n, \cdot, \cdot)$  is Hermitian linear<sup>5</sup>. We say that  $\mathbf{t}$  is positive definite if

$$\sum_{k,l} \mathbf{t}(m_k, m_l, \mathbf{a}_k, \mathbf{a}_l) \geq 0 \quad \text{for any finite choice of sequences } (m_k)_k \text{ and } (\mathbf{a}_k)_k, \quad (3)$$

Our main result can be summarised as follows.

QUINTESSANCE. *Given  $(s(m, n))_{m, n=0}^{\infty}$ , satisfying the condition*

$$s(m+3, n) - 3s(m+2, n+1) + 3s(m+1, n+2) - s(m, n+3) = 0, \quad m, n \in \mathbb{N}. \quad (4)$$

*there exists a form  $\mathbf{s}$  over  $\mathbb{N}^2 \times \mathbb{C}^4$  which is translation invariant, i.e. satisfies*

$$\mathbf{s}(m+k, n, \mathbf{a}, \mathbf{b}) = \mathbf{s}(m, n+k, \mathbf{a}, \mathbf{b}), \quad m, n, k = 0, 1, \dots \quad (5)$$

*and is in the following relation with  $s$*

$$\begin{aligned} s(m, n) &= \mathbf{s}(m, n, \mathbf{e}_0, \mathbf{e}_0) + m\mathbf{s}(m-1, n, \mathbf{e}_1, \mathbf{e}_0) \\ &+ n\mathbf{s}(m, n-1, \mathbf{e}_0, \mathbf{e}_1) + mn\mathbf{s}(m-1, n-1, \mathbf{e}_1, \mathbf{e}_1), \quad m, n \in \mathbb{N}, \end{aligned} \quad (6)$$

*with*

$$\mathbf{e}_0 \stackrel{\text{def}}{=} (1, 0) \text{ and } \mathbf{e}_1 \stackrel{\text{def}}{=} (0, 1) \quad (7)$$

*standing for the basic vectors in  $\mathbb{C}^2$  and a convention that whenever a negative argument appears the corresponding entry is valued to 0. Furthermore, if  $\mathbf{s}$  is additionally positive definite then  $s$  is a Sobolev moment bisequence.*

*Conversely, for every Sobolev moment bisequence the condition (4) holds, the form*

$$\mathbf{s}(m, n, \mathbf{a}, \mathbf{b}) = \sum_{i,j=0}^1 a_i \bar{b}_j \int_{\mathbb{R}} t^{m+n} d\mu_{i,j}, \quad \mathbf{a} = (a_0, a_1), \quad \mathbf{b} = (b_0, b_1), \quad (8)$$

<sup>3</sup> It can be considered alternatively as an infinite matrix; we abandon this point of view in order to avoid any confusion.

<sup>4</sup> This is a cosine moment problem, not a power one like those commonly considered, so it “continuous” in a sense.

<sup>5</sup> The equivalent term ‘sesqui-linear’ appears often in the literature.

satisfies (6) and is translation invariant and positive definite.

Formula (4) comes from [2, condition (1.2a), p. 309], consequently we will call it the *Helton condition*. Its operator theoretic master can be found in [2], however the circumstances there are rather limited.

*Remark 1.* As we show later, every Sobolev moment sequence has the form

$$s(m, n) = \langle (T + N)^m V(1, 0), (T + N)^n V(1, 0) \rangle_{\mathcal{K}}, \quad m, n = 0, 1, \dots$$

where  $V: \mathbb{C}^2 \rightarrow \mathcal{K}$  is an isometry,  $T$  is selfadjoint and  $N$  is a nilpotent operator in some Hilbert space  $\mathcal{K}$ , see equations (28) and (29), which can be viewed as the Jordan dilation of  $s$ .

**2. The Sobolev moment problem; necessary conditions.** First we argue that the conditions appearing in Quintessence are necessary, i.e. we show the 'Conversely' part of it. Suppose that  $(s(m, n))_{m, n=0}^\infty$  is a Sobolev moment bisequence. The integral representation (2) suggests the following decomposition (whenever a negative argument appears the corresponding entry is valued to 0)

$$\begin{aligned} s(m, n) &= s_{0,0}(m, n) + ms_{0,1}(m-1, n) \\ &\quad + ns_{1,0}(m, n-1) + (m-1)(n-1)s_{1,1}(m-1, n-1) \end{aligned} \quad (9)$$

as well as the properties

$$\begin{aligned} s_{0,0}(m, n) &= s_{0,0}(m+n, 0) = s_{0,0}(0, m+n) \\ s_{0,1}(m, n) &= s_{0,1}(m+n, 0) = s_{0,1}(0, m+n) \\ s_{1,0}(m, n) &= s_{1,0}(m+n, 0) = s_{1,0}(0, m+n) \\ s_{1,1}(m, n) &= s_{1,1}(m+n, 0) = s_{1,1}(0, m+n). \end{aligned} \quad (10)$$

If a bisequence  $s$  satisfies the above properties (9) and (10) then for  $\mathbf{a} = (a_0, a_1), \mathbf{b} = (b_0, b_1) \in \mathbb{C}^2$  and  $m, n \in \mathbb{N}$  consider

$$s(m, n, \mathbf{a}, \mathbf{b}) \stackrel{\text{def}}{=} a_0 \bar{b}_0 s_{0,0}(m, n) + a_0 \bar{b}_1 s_{0,1}(m, n) + a_1 \bar{b}_0 s_{1,0}(m, n) + a_1 \bar{b}_1 s_{1,1}(m, n). \quad (11)$$

Consequently  $\mathbf{s}$  is a form over  $\mathbb{N}^2 \times \mathbb{C}^4$ . Conditions (10) are equivalent to translational invariance of  $\mathbf{s}$  on  $\mathbb{N} \times \mathbb{N}$ , that is to

$$\mathbf{s}(m, n, \mathbf{a}, \mathbf{b}) = \mathbf{s}(m+n, 0, \mathbf{a}, \mathbf{b}) = \mathbf{s}(0, m+n, \mathbf{a}, \mathbf{b}), \quad m, n = 0, 1, \dots \quad (12)$$

which is the same as (5). The form  $\mathbf{s}$  satisfies (6), which can be easily checked by direct calculation. To finish the study of the necessary conditions, we need to show the following statement.

**PROPOSITION 2.** *If  $(s(m, n))_{m, n=0}^\infty$  is a Sobolev moment bisequence and we define<sup>6</sup>*

$$s_{i,j}(m, n) \stackrel{\text{def}}{=} \int_{\mathbb{R}} t^{m+n} \mu_{i,j}(dt), \quad i, j = 0, 1 \quad (13)$$

*then  $\mathbf{s}$  defined by (11) is a positive definite form.*

Before proving it let us recall some necessary background. Note that if  $s$  is a Sobolev moment sequence, the measure  $\mu$  can be always viewed as a semispectral<sup>7</sup> one in  $\mathbb{C}^2$  (cf. [6, Theorem 4 p. 30]), i.e. a  $2 \times 2$ -matrix valued function defined on Borel subsets of  $\mathbb{R}$ , which is countably additive (each of the two possibilities: in strong and weak operator topology, in this case is equivalent in this case to the entry-wise convergence). A semispectral measure becomes spectral if it is orthogonal projection valued.

<sup>6</sup> Given a Sobolev moment sequence the decomposition into four translation invariant sequences (9), (10) may not be unique, as will be seen later on, cf. Proof of Theorem 5, formulas (15) and (16) therein. Therefore, in formula (13) we need to refer to a specific decomposition explicitly.

<sup>7</sup> It is sometimes referred to as POV(=positive operator valued) measure.

THEOREM 3 (cf. [6], p. 30). *A  $2 \times 2$  matrix  $(\mu_{i,j})_{i,j=0}^1$  of measures on  $\mathbb{R}$  is positive definite if and only if there is a Hilbert space  $\mathcal{K}$ , and a bounded linear operator  $R: \mathbb{C}^2 \rightarrow \mathcal{K}$  and a spectral measure  $E$  in  $\mathcal{K}$  such that*

$$(\mu_{i,j}(\rho))_{i,j=0}^1 = R^* E(\rho) R, \quad \rho \text{ a Borel subset of } \mathbb{R}. \quad (14)$$

Remark 4. In general  $R^* R = (\mu_{i,j}(\mathbb{R}))_{i,j=0}^1$ ; in particular,  $R^*$  is the orthogonal projection of  $\mathcal{K}$  onto  $\mathbb{C}^2$  if and only if  $(\mu_{i,j}(\mathbb{R}))_{i,j=0}^1 = \text{diag}(1, 1)$ .

Notice that (14) is equivalent to

$$\mu_{i,j}(\cdot) = \langle E(\cdot) R e_i, R e_j \rangle, \quad i, j = 0, 1.$$

PROOF OF PROPOSITION 2. With  $\mathbf{a}_k = (a_{0,k}, a_{1,k})$  and  $\mathcal{K}$ ,  $E$  and  $R$  as above, one has

$$\begin{aligned} \sum_{k,l} s(m_k, m_l, \mathbf{a}_k, \mathbf{a}_l) &= \sum_{k,l} \sum_{i,j=0}^1 \int_{\mathbb{R}} a_{i,k} t^{m_k^{(i)}} \bar{a}_{j,l} t^{m_l^{(j)}} \mu_{i,j}(dt) \\ &= \sum_{i,j=0}^1 \int_{\mathbb{R}} \left( \sum_k a_{i,k} t^{m_k^{(i)}} \right) \left( \sum_l \bar{a}_{j,l} t^{m_l^{(j)}} \right) \mu_{i,j}(dt) \\ &= \sum_{i,j=0}^1 \int_{\mathbb{R}} \left( \sum_k a_{i,k} t^{m_k^{(i)}} \right) \left( \sum_n \bar{a}_{j,l} t^{m_l^{(j)}} \right) \langle E(dt) R e_i, R e_j \rangle_{\mathcal{K}} \\ &= \sum_{i,j=0}^1 \left\langle \int_{\mathbb{R}} \left( \sum_k a_{i,k} t^{m_k^{(i)}} \right) \left( \sum_l \bar{a}_{j,l} t^{m_l^{(j)}} \right) E(dt) R e_i, R e_j \right\rangle_{\mathcal{K}} \\ &\stackrel{s}{=} \int_{\mathbb{R}} \left\| \sum_{i=0}^1 \sum_k a_{i,k} t^{m_k^{(i)}} E(dt) R e_i \right\|_{\mathcal{K}}^2 \geq 0, \end{aligned}$$

where in the passage  $\stackrel{s}{=}$  multiplicativity of the spectral integral is used.  $\square$

### 3. More on Helton's condition. We prove one of our basic results.

THEOREM 5. *Given a bisequence  $(s(m, n))_{m,n=0}^\infty$ , the following conditions are equivalent*

- (h1) *s satisfies the Helton condition (4);*
- (h2) *there are four bisequences  $(s_{i,j}(m, n))_{m,n=0}^\infty$ ,  $i, j = 0, 1$  with the translation invariance properties (10) and such that the bisequence s decomposes as in (9);*
- (h3) *there exists a form s with the translation invariance property (5), satisfying (6).*

PROOF. The implication (h3) $\Rightarrow$ (h2) is trivial, the reverse implication was proved in the previous section, see formula (11). To show (h2) $\Rightarrow$ (h1) it is enough to evaluate the left hand side of (4), which after using we use (9) and (10) and grouping of the summands results in getting 0. The main part of the proof is the implication (h1) $\Rightarrow$ (h2).

Assume that (h1) is satisfied. Let  $\mathbb{C}[\mathbb{N}]$  stands for the linear space of of complex finitely supported functions on  $\mathbb{N}$  considered point-wisely and  $\mathbb{C}[\mathbb{N} \times \mathbb{N}]$  that of finitely supported functions on  $[\mathbb{N} \times \mathbb{N}]$ . The Kronecker deltas  $\delta_n$  and  $\delta_{m,n}$  are apparently members of  $\mathbb{C}[\mathbb{N}]$  and  $\mathbb{C}[\mathbb{N} \times \mathbb{N}]$ , respectively. We naturally embed  $\mathbb{N}$  in  $\mathbb{C}[\mathbb{N}]$  as  $n \mapsto \delta_n$ ; therefore  $a \in \mathbb{C}[\mathbb{N}]$  can be written as  $\sum_{n \in \mathbb{N}} a(n) \delta_n$ ; for technical reasons set  $\delta_{-1} = \delta_{-2} = 0$ .

The bisequence  $s \stackrel{\text{def}}{=} (s(m, n))_{m,n=0}^\infty$  determines the unique linear mapping

$$s^\dagger : \mathbb{C}[\mathbb{N} \times \mathbb{N}] \rightarrow \mathbb{C}$$

by

$$s^\dagger(a) \stackrel{\text{def}}{=} \sum_{m,n \in \mathbb{N}} a(m, n) s(m, n), \quad a \in \mathbb{C}[\mathbb{N} \times \mathbb{N}].$$

Clearly

$$s(m, n) = s^\dagger(\delta_{m,n}).$$

Moreover if we consider  $\mathbb{C}[\mathbb{N}]$  as the commutative algebra with the standard convolution

$$(a * b)(n) = \sum_{x+y=n} a(x)b(y), \quad a, b \in \mathbb{C}[\mathbb{N}]$$

the condition (4) turns out to be equivalent to

$$s^\dagger(p_1 * \delta_{m,n}) = 0, \quad m, n \in \mathbb{N},$$

with

$$p_1 = \delta_{3,0} - 3\delta_{2,1} + 3\delta_{1,2} - \delta_{0,3}.$$

We define a linear mapping

$$\Gamma : \mathbb{C}[\mathbb{N} \times \mathbb{N}] \rightarrow \mathbb{C}^{2 \times 2}[\mathbb{N}],$$

by

$$\Gamma(\delta_{m,n}) = \begin{pmatrix} \delta_{m+n} & m\delta_{m+n-1} \\ n\delta_{m+n-1} & mn\delta_{m+n-2} \end{pmatrix}, \quad m, n \in \mathbb{N}.$$

Let us now get back to the assumption (4). Note that it can be rewritten as

$$\mathfrak{I}(p_1) \subseteq \ker s^\dagger, \quad (15)$$

where  $\mathfrak{I}(p_1)$  denotes the ideal generated by  $p_1$ . We will show (16) later on that

$$\mathfrak{I}(p_1) = \ker \Gamma, \quad (16)$$

now observe that this will finish the proof. Indeed, the fundamental homomorphism theorem together with (15) and (16) implies that there exists a linear mapping

$$\Phi : \mathbb{C}^{2 \times 2}[\mathbb{N}] \rightarrow \mathbb{C},$$

satisfying

$$\Phi \circ \Gamma = s^\dagger.$$

Writing this equality on  $\delta_{m,n}$  we get

$$\begin{aligned} s(m, n) &= s^\dagger(\delta_{m,n}) = \Phi \begin{pmatrix} \delta_{m+n} & m\delta_{m+n-1} \\ n\delta_{m+n-1} & mn\delta_{m+n-2} \end{pmatrix} \\ &= \Phi \begin{pmatrix} \delta_{m+n} & 0 \\ 0 & 0 \end{pmatrix} + m\Phi \begin{pmatrix} 0 & \delta_{m+n-1} \\ 0 & 0 \end{pmatrix} + n\Phi \begin{pmatrix} 0 & 0 \\ \delta_{m+n-1} & 0 \end{pmatrix} + mn\Phi \begin{pmatrix} 0 & 0 \\ 0 & \delta_{m+n-2} \end{pmatrix} \end{aligned}$$

which gives the desired representation (9) and the relation (10).

Hence, to finish the proof of Theorem 5 it is enough to show that (16) holds. The inclusion  $\subseteq$  follows by direct computation. To prove (16) it remains to show the converse  $\ker \Gamma \subseteq \mathfrak{I}(p_1)$ . Let  $a \in \ker \Gamma$  which means

$$\begin{aligned} \sum_{m,n} a(m, n) \delta_{m+n} &= 0, \quad \sum_{m,n} a(m, n) m \delta_{m+n-1} = 0, \\ \sum_{m,n} a(m, n) n \delta_{m+n-1} &= 0, \quad \sum_{m,n} a(m, n) mn \delta_{m+n-2} = 0, \end{aligned} \quad (17)$$

with the convention that  $\delta_{-1} = \delta_{-2} = 0$ .

We show now that (17) implies  $a \in \mathfrak{I}(p_1)$ . The following decomposition holds

$$a = \sum_{n \in \mathbb{N}} a^n, \quad a^n \stackrel{\text{def}}{=} \sum_{i=0}^n a(i, n-i) \delta_{i, n-i}, \quad n \in \mathbb{N}$$

with the first sum to be terminating. Note that each  $a^n$  ( $n \in \mathbb{N}$ ) enjoys the properties (17) as well. Hence, without any loss of generality we may assume that

$$a = \sum_{i=0}^n a_i \delta_{i, n-i},$$

for which formulae (17) reduce to

$$\sum_{i=0}^n a_i = 0, \quad \sum_{i=0}^n i \cdot a_i = 0, \quad \sum_{i=0}^n (n-i) \cdot a_i = 0, \quad \sum_{i=0}^n i(n-i) \cdot a_i = 0.$$

Observe, that the second of these equations is a consequence of the first and the third and we may skip it. The remaining equations can be written in the equivalent form

$$(a_0, \dots, a_n)^\top \in \ker X_n,$$

where the matrix  $X_n$  is given by

$$X_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ n & n-1 & (n-2) & \cdots & 1 & 0 \\ 0 \cdot n & 1 \cdot (n-1) & 2 \cdot (n-2) & \cdots & (n-1) \cdot 1 & n \cdot 0 \end{pmatrix}.$$

Note that for  $n = 0, 1$  the matrix  $X_n$  has a trivial kernel. For  $n \geq 2$  the first and the last two columns of  $X_n$  are clearly linearly independent and hence

$$\dim \ker X_n = (n+1) - 3 = n-2, \quad n \geq 3. \quad (18)$$

We will show simultaneously by induction with respect to  $n \in \mathbb{N}$ ,  $n \geq 3$  that

- (i) if  $(a_0, \dots, a_n)^\top \in \ker X_n$  is a solution of the above system of equations then  $\sum_{i=0}^n a_i \delta_{i, n-i} \in \mathcal{I}(p_1)$ ,
- (ii) there exists  $(a_0, \dots, a_n)^\top \in \ker X_n$  with  $a_0 \neq 0$ ,

which will finish the proof. The kernel of  $X_3$  is spanned by  $(a_0, a_1, a_2, a_3)^\top = (-1, 3, -3, 1)^\top$ , hence  $a = p_1$  and in consequence (i) and (ii) are satisfied. Now suppose that both (i) and (ii) are true for some  $n \geq 3$ . First observe that if  $y \in \ker X_n$  then  $(0, y)^\top \in \ker X_{n+1}$ . Indeed, subtracting the second row of  $X_{n+1}$  from the third one obtains the matrix

$$\left( \begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 & 1 \\ n+1 & n & (n-1) & \cdots & 1 & 0 \\ -n-1 & 0 & (n-1) & \cdots & (n-1) \cdots 1 & 0 \end{array} \right) = \left( \begin{array}{c|c} 1 & \\ n+1 & X_n \\ -n-1 & \end{array} \right).$$

Consequently, by (18), there are  $n-2$  linearly independent vectors in  $\ker X_{n+1}$  of the form  $(0, a_1, a_2, \dots, a_{n+1})^\top$ . Note that for all these vectors one has

$$a = \sum_{i=1}^{n+1} a_i \delta_{i, n+1-i} = \delta_{1,0} * \sum_{i=0}^n a_{i+1} \delta_{i, n-i}.$$

By induction, the element

$$\sum_{i=0}^n a_{i+1} \delta_{i, n-i}$$

is in  $\mathcal{I}(p_1)$ , as  $(a_1, \dots, a_{n+1})^\top \in \ker X_n$ . Hence,  $a \in \mathcal{I}(p_1)$ . In view of (18) (with  $n$  replaced by  $n+1$ ), to finish the induction step it is enough to show that there exists a vector  $(a_0, a_1, \dots, a_{n+1})^\top \in \ker X_{n+1}$  with  $a_0 \neq 0$  and  $a = \sum_{i=0}^{n+1} a_i \delta_{i, n+1-i} \in \mathcal{I}(p_1)$ . By the induction assumption (ii) there exists  $(b_0, \dots, b_n) \in \ker X_n$  with  $b_0 \neq 0$ . We define  $b_{-1} = b_{n+1} = 0$  and  $a_i = b_{i-1} - b_i$ ,  $i = 0, \dots, n+1$ . Since  $b = \sum_{i=1}^n b_i \delta_{i, n-i} \in \mathcal{I}(p_1)$ , we have

$$a = \sum_{i=0}^{n+1} a_i \delta_{i, n+1-i} = p_0 * b \in \mathcal{I}(p_1), \quad p_0 = \delta_{1,0} - \delta_{0,1}.$$

Since  $a_0 = -b_0 \neq 0$ , the vector  $(a_0, a_1, \dots, a_{n+1})^\top$  is linearly independent from vectors of the form  $(0, y)^\top$  with  $y \in \ker X_n$ , which finishes the induction step.  $\square$

Comparing what we proved so far with Quintessence in Introduction we see that what remains to show is that the positive definiteness of  $\mathbf{s}$  implies existence of the measures  $(\mu_{ij})_{ij=0}^1$ . This turns in considering a moment problem of Hamburger type for the  $2 \times 2$  matrix  $(\mathbf{s}(m, n, \mathbf{a}, \mathbf{b}))_{m, n=0}^\infty$ . One of the ways of solving this problem is to employ dilation theory, as understood in [10, 13]. This can be conveniently carried out in the RKHS environment.

**4. The RKHS buildup.** As mentioned earlier positive definiteness (3) makes it possible to introduce the reproducing kernel Hilbert space and consider operators therein. Define the sections  $\mathbf{s}_{(n, \mathbf{b})}$  and their linear span  $\mathcal{D}$  as

$$\mathbf{s}_{(n, \mathbf{b})} \stackrel{\text{def}}{=} \mathbf{s}(\cdot, n, -, \mathbf{b}), \quad \mathcal{D} \stackrel{\text{def}}{=} \text{lin}\{\mathbf{s}_{(n, \mathbf{b})} : (n, \mathbf{b}) \in \mathbb{N} \times \mathbb{C}^2\} \quad (19)$$

and equipped  $\mathcal{D}$  with an inner product extended from<sup>8</sup>

$$\langle \mathbf{s}_{(n, \mathbf{b})}, \mathbf{s}_{(m, \mathbf{a})} \rangle \stackrel{\text{def}}{=} \mathbf{s}(m, n, \mathbf{a}, \mathbf{b}). \quad (20)$$

The reproducing kernel Hilbert space  $\mathcal{H}$  determined by (20) is composed of complex valued functions on  $\mathbb{N} \times \mathbb{C}^2$  whereas the reproducing property (restricted to  $\mathcal{D}$  which is a dense subspace of  $\mathcal{H}$ ) is precisely

$$\mathbf{s}_{(n, \mathbf{b})}(m, \mathbf{a}) = \langle \mathbf{s}_{(n, \mathbf{b})}, \mathbf{s}_{(m, \mathbf{a})} \rangle, \quad (21)$$

which is just the definition of the inner product read *à rebours*.

Fix  $k \in \mathbb{N}$  and set for  $(n, \mathbf{b}) \in \mathbb{N} \times \mathbb{C}^2$

$$(\Phi(k)\mathbf{s}_{(n, \mathbf{b})})(m, \mathbf{a}) \stackrel{\text{def}}{=} \mathbf{s}_{(n, \mathbf{b})}(k + m, \mathbf{a}), \quad (m, \mathbf{a}) \in \mathbb{N} \times \mathbb{C}^2 \quad (22)$$

$$\Psi(k)\mathbf{s}_{(n, \mathbf{b})} \stackrel{\text{def}}{=} \mathbf{s}_{(n+k, \mathbf{b})} \quad (23)$$

defined so far for  $\mathbf{s}_{(n, \mathbf{b})}$ 's. Notice  $\Psi(k)$  extends linearly to the whole of  $\mathcal{D}$  as an operator.

Fortunately, and exclusively in this case,

$$\begin{aligned} \langle \mathbf{s}_{(m, \mathbf{a})}, \Psi(k)\mathbf{s}_{(n, \mathbf{b})} \rangle &\stackrel{(23)}{=} \langle \mathbf{s}_{(m, \mathbf{a})}, \mathbf{s}_{(n+k, \mathbf{b})} \rangle \stackrel{(21)}{=} \mathbf{s}_{(m, \mathbf{a})}(n+k, \mathbf{b}) \\ &\stackrel{(19)}{=} \mathbf{s}(n+k, m, \mathbf{b}, \mathbf{a}) \stackrel{(5)}{=} \mathbf{s}(n, m+k, \mathbf{b}, \mathbf{a}) \\ &\stackrel{(19)}{=} \mathbf{s}_{(m+k, \mathbf{a})}(n, \mathbf{b}) \stackrel{(22)}{=} \Psi(k)\mathbf{s}_{(m, \mathbf{a})}(n, \mathbf{b}) \\ &\stackrel{(21)}{=} \langle \Psi(k)\mathbf{s}_{(m, \mathbf{a})}, \mathbf{s}_{(n, \mathbf{b})} \rangle, \end{aligned}$$

which implies  $\Psi(k)$  is symmetric. Furthermore

$$\Phi(k)\mathbf{s}_{(n, \mathbf{b})}(m, \mathbf{a}) \stackrel{(23)}{=} \mathbf{s}_{(n, \mathbf{b})}(k+m, \mathbf{a}) \stackrel{(19)}{=} \mathbf{s}_{(n+k, \mathbf{b})}(m, \mathbf{a}) \stackrel{(23)}{=} \Psi(k)\mathbf{s}_{(n, \mathbf{b})}(m, \mathbf{a}). \quad (24)$$

which means  $\Phi(k) = \Psi(k)$  on  $\mathcal{D}$ . As a consequence  $\Phi(k)$  extends to a linear operator as well.

Both,  $\Phi(k)$  and  $\Psi(k)$  are the standard, naturally defined operators in a RKHS.

**5. Jordan operators in the RKHS  $\mathcal{H}$ .** With notation (7) in mind consider an operator  $N$  defined on  $\mathcal{D}$  as the linear extension of the formula

$$N\mathbf{s}_{(n, \mathbf{b})} \stackrel{\text{def}}{=} \bar{b}_0 \mathbf{s}_{(n, \mathbf{e}_1)}.$$

Set  $T \stackrel{\text{def}}{=} \Phi(1)$ ; because the mapping  $k \rightarrow \Phi(k)$  is a semigroup homomorphism on  $\mathbb{N}$  to linear operators on  $\mathcal{D}$  we have  $T^k = \Phi(k)$ . Apparently,

$$N \text{ commutes with } T \text{ on } \mathcal{D} \quad (25)$$

and

$$N^2 = 0 \text{ on } \mathcal{D}, \text{ that is } N \text{ is nilpotent.} \quad (26)$$

---

<sup>8</sup> Notice  $\mathbf{s}_{(n, \mathbf{b})}$  is antilinear in  $\mathbf{b}$ .

Due to (25) and (26), the (commutative) Newton's binomial formula is applicable and reduces to

$$(T + N)^k = T^k + kT^{k-1}N \text{ on } \mathcal{D}. \quad (27)$$

This motivates the following state of affair. Given an arbitrary Hilbert space  $\mathcal{K}$ , say<sup>9</sup> call an operator of the form  $T + N$  where  $T$  is symmetric and  $N$  is such that  $N^2 = 0$ , and  $T$  and  $N$  commute on a dense subspace  $\mathcal{D}$  of  $\mathcal{K}$ , invariant for both and being a core of each of them, a *Jordan operator of order 2*; the case when both of  $T$  and  $N$  are bounded is considered in [2]. Furthermore, call the couple  $T$  and  $N$  a *Jordan dilation of  $s$  relative to the decomposition (9)* if

$$s(m, n) = \langle (T + N)^m V(1, 0), (T + N)^n V(1, 0) \rangle_{\mathcal{K}}, \quad m, n = 0, 1, \dots \quad (28)$$

where  $V: \mathbb{C}^2 \rightarrow \mathcal{D}$  is an isometry, and

$$s(l, k, \mathbf{b}, \mathbf{a}) = \langle T^k V \mathbf{b}, T^l V \mathbf{a} \rangle, \quad k, l \in \mathbb{N}, \mathbf{a}, \mathbf{b} \in \mathbb{C}^2. \quad (29)$$

## 6. The basic result.

**THEOREM 6.** *Given  $(s(m, n))_{m, n=0}^\infty$ . The following conditions are equivalent:*

- (i) *the condition (h1) is satisfied and  $s$  defined by (11) and (13) is a positive definite form;*
- (ii) *the condition (h1) is satisfied and there is a Jordan dilation  $T, N$  of the bisequence  $s$  relative to the decomposition (9);*
- (iii)  *$s$  is a Sobolev moment bisequence.*

**PROOF.** Most of the arguments have been already presented in the constructions above. Let us put them together.

(i)  $\implies$  (ii). Use the constructions done in sections 4 and 5 maintaining the notation applied there. Consider the standard embedding

$$V: \mathbb{C}^2 \ni \mathbf{a} \mapsto s_{0, \mathbf{a}} \in \mathcal{H}.$$

It is a matter of straightforward though lengthy verification that

$$s(m, n) = \langle (T + N)^m V(1, 0), (T + N)^n V(1, 0) \rangle_{\mathcal{H}}, \quad m, n = 0, 1, \dots \quad (30)$$

which is nothing but (28) under the specific circumstances considered there.

Indeed, due to (27), (9) and (20),

$$\begin{aligned} & \langle (T + N)^m V(1, 0), (T + N)^n V(1, 0) \rangle \\ &= \langle (T^m + mT^{m-1}N)V(1, 0), (T^n + nT^{n-1}N)V(1, 0) \rangle \\ &= \langle T^m s_{(0, \mathbf{e}_0)}, T^n s_{(0, \mathbf{e}_0)} \rangle + n \langle T^m s_{(0, \mathbf{e}_0)}, T^{n-1} s_{(0, \mathbf{e}_1)} \rangle + m \langle T^{m-1} s_{(0, \mathbf{e}_0)}, T^n s_{(0, \mathbf{e}_1)} \rangle \\ & \quad + mn \langle T^{m-1} s_{0, \mathbf{e}_1}, T^{n-1} s_{0, \mathbf{e}_1} \rangle \\ &= \langle s_{(m, \mathbf{e}_0)}, s_{(n, \mathbf{e}_0)} \rangle + n \langle s_{(m, \mathbf{e}_0)}, s_{(n-1, \mathbf{e}_1)} \rangle + m \langle s_{(n-1, \mathbf{e}_0)}, s_{(0, \mathbf{e}_1)} \rangle + mn \langle s_{m-1, \mathbf{e}_1}, s_{n-1, \mathbf{e}_1} \rangle \\ &= s(n, m, \mathbf{e}_0, \mathbf{e}_0) + ns(n-1, m, \mathbf{e}_1, \mathbf{e}_0) + ms(n, m-1, \mathbf{e}_0, \mathbf{e}_1) + mns(n-1, m-1, \mathbf{e}_1, \mathbf{e}_1) \\ &= s_{0,0}(m, n) + ms_{0,1}(m-1, n) + ns_{1,0}(m, n-1) + mns_{1,1}(m-1, n-1) = s(m, n). \end{aligned}$$

(ii)  $\implies$  (iii). Because  $V^*$  is a projection onto a 2-dimensional space  $T$  can be represented by  $2 \times 2$ -matrix valued semispectral measure  $\mu$  (which is a compression<sup>10</sup> to the

<sup>9</sup> Notice now the space may be not be the same as the above hence the notation is different

<sup>10</sup> This is when the formula (14) is read "right-to-left" with  $R$  being an isometry.



2-dimensional space of a spectral measure of any selfadjoint extension<sup>11</sup> of  $T$ ) as

$$\mathbf{s}(l, k, \mathbf{b}, \mathbf{a}) = \langle T^k V \mathbf{b}, T^l V \mathbf{a} \rangle = \int_{\mathbb{R}} t^{k+l} \langle \mu(dt) \mathbf{b}, \mathbf{a} \rangle. \quad (31)$$

Because  $V \mathbf{e}_i \perp V \mathbf{e}_j$  ( $V$  is an isometry!) if  $i \neq j$  and  $\text{lin}(\mathbf{s}_{(n, \mathbf{b})})_{(n, \mathbf{b})}$  is invariant for  $T$ , we have the decomposition

$$\begin{pmatrix} \mu_{0,0}(\cdot) & \mu_{0,1}(\cdot) \\ \mu_{1,0}(\cdot) & \mu_{1,1}(\cdot) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \langle \mu(\cdot) \mathbf{e}_0, \mathbf{e}_0 \rangle & \langle \mu(\cdot) \mathbf{e}_0, \mathbf{e}_1 \rangle \\ \langle \mu(\cdot) \mathbf{e}_1, \mathbf{e}_0 \rangle & \langle \mu(\cdot) \mathbf{e}_1, \mathbf{e}_1 \rangle \end{pmatrix} \quad (32)$$

Again, with some effort,

$$\begin{aligned} & \langle (T + N)^m V(1, 0), (T + N)^n V(1, 0) \rangle \\ &= \langle T^m \mathbf{s}_{(0, \mathbf{e}_0)}, T^n \mathbf{s}_{(0, \mathbf{e}_0)} \rangle + n \langle T^m \mathbf{s}_{(0, \mathbf{e}_0)}, T^{n-1} \mathbf{s}_{(0, \mathbf{e}_1)} \rangle + m \langle T^{m-1} \mathbf{s}_{(0, \mathbf{e}_0)}, T^n \mathbf{s}_{(0, \mathbf{e}_1)} \rangle \\ & \quad + mn \langle T^{m-1} \mathbf{s}_{0, \mathbf{e}_1}, T^{n-1} \mathbf{s}_{0, \mathbf{e}_1} \rangle \\ &= \mathbf{s}(n, m, \mathbf{e}_0, \mathbf{e}_0) + n \mathbf{s}(n-1, m, \mathbf{e}_1, \mathbf{e}_0) + m \mathbf{s}(m-1, n, \mathbf{e}_0, \mathbf{e}_1) + mn \mathbf{s}(m-1, n-1, \mathbf{e}_1, \mathbf{e}_1) \\ & \stackrel{(31)(32)}{=} \int_{\mathbb{R}} t^m t^n \mu_{0,0}(dt) + n \int_{\mathbb{R}} t^m t^{n-1} \mu_{0,1}(dt) + m \int_{\mathbb{R}} t^{m-1} t^n \mu_{1,0}(dt) \\ & \quad + mn \int_{\mathbb{R}} t^{m-1} t^{n-1} \mu_{1,1}(dt) \end{aligned}$$

which confronted with (30) establishes

$$s(m, n) = \sum_{i,j=0}^1 \int_{\mathbb{R}} t^{m(i)} t^{n(j)} \mu_{i,j}(dt), \quad m, n = 0, 1, \dots$$

(iii)  $\implies$  (i). This statement was proved as Proposition 2.  $\square$

**7. Compactly supported representing measures  $\mu$ .** For a measure  $\mu$  such that (31) holds to be compactly supported it is necessary and sufficient the operator  $T$  therein to be bounded. For that there is a number of equivalent conditions in terms of  $\mathbf{s}$ , which are applicable here due to the fact that  $\mathbf{s}$  is a translationally invariant form in the sense of (12) or (5). The conditions are listed in [16, Lemma 2] and are originated in [11, 12, 13]. Let us pick up some of them explicitly.

**PROPOSITION 7.** *The following conditions are equivalent to the boundedness of  $T$*

- for every  $k = 0, 1, \dots$  there is  $d(k)$  such that

$$\mathbf{s}(m+k, m+k, \mathbf{a}, \mathbf{a}) \leq d(k) \mathbf{s}(m, m, \mathbf{a}, \mathbf{a}), \quad m \in \mathbb{N}, \mathbf{a} \in \mathbb{C}^2;$$

- for every  $k = 0, 1, \dots$  there is a function  $\alpha: \mathbb{N} \rightarrow [0, +\infty)$  such that  $\alpha(k+l) \leq \alpha(k)\alpha(l)$ ,  $k, l = 0, 1, \dots$  for which

$$|\mathbf{s}(m, m, \mathbf{a}, \mathbf{a})| \leq c(\mathbf{a}) \alpha(m), \quad m \in \mathbb{N}, \mathbf{a} \in \mathbb{C}^2;$$

- For any  $k = 0, 1, \dots$  and a finite choice of  $m_i$ 's and  $\mathbf{a}_i$ 's

$$\liminf_{l \rightarrow +\infty} \sum_{i,j} \mathbf{s}(m_i + lk, m_j + kl, \mathbf{a}_i, \mathbf{a}_j)^{2^{-l}} < +\infty$$

<sup>11</sup> Selfadjoint extensions may be quite diverse, often not necessarily unitary equivalent, cf. [1] and also [17]. In our case apparently they can be obtained as von Neumann extensions, as those acting in the same space, and also as Naimark extension, which go beyond the space. The latter applies also to the case of equal deficiency indices. In this way we may get plenty of representing measures  $\mu$ , cf. [1, 17].

**8. Defect indices of  $T$ .** The following statement is true.

**THEOREM 8.** *The operator  $T$ , defined as in Section 5, has equal defect indices, not larger than 2.*

**PROOF.** We define a conjugation (anti-linear) operator  $J$  on  $\mathcal{D}$  by

$$J \left( \sum_{n,\mathbf{a}} \alpha_{n,\mathbf{a}} \mathbf{s}_{n,\mathbf{a}} \right) = \sum_{n,\mathbf{a}} \bar{\alpha}_{n,\mathbf{a}} \mathbf{s}_{n,\mathbf{a}}.$$

A standard reproducing kernel Hilbert space argument shows that  $J$  is properly defined, preserves the norm and extends uniquely to the whole space  $\mathcal{H}$ . Furthermore,  $JT = TJ$  and hence, by the von Neumann theorem, the defect indices of  $T$  are equal.

Now let us take any element  $f \in \mathcal{N}(T^* - z)$ , with  $\text{Im} z > 0$ . Note that

$$\mathbf{s}_{(n,\mathbf{a})} = \bar{a}_0 \mathbf{s}_{(n,\mathbf{e}_0)} + \bar{a}_1 \mathbf{s}_{(n,\mathbf{e}_1)}, \quad \mathbf{a} = (a_0, a_1),$$

which can be easily checked by taking the inner product of the both sides with an arbitrary kernel function  $\mathbf{s}_{(m,b)}$  and using the fact that  $\mathbf{s}$  is a form. Consequently,

$$\begin{aligned} f(n, \mathbf{a}) &= \langle f, \mathbf{s}_{(n,\mathbf{a})} \rangle = \langle f, T^n \mathbf{s}_{(0,\mathbf{a})} \rangle = \langle \bar{z}^n f, \mathbf{s}_{(0,\mathbf{a})} \rangle \\ &= a_0 \langle \bar{z}^n f, \mathbf{s}_{(0,\mathbf{e}_0)} \rangle + a_1 \langle \bar{z}^n f, \mathbf{s}_{(0,\mathbf{e}_1)} \rangle \\ &= a_0 \bar{z}^n f(0, \mathbf{e}_0) + a_1 \bar{z}^n f(0, \mathbf{e}_1). \end{aligned}$$

This means that the kernel  $\mathcal{N}(T^* - z)$  is of dimension at most two.  $\square$

We present now an example, where the defect indices of  $T$  are indeed  $(2, 2)$ .

*Example 9.* Consider an indeterminate measure  $\mu$  on the real line, set  $\mu_{i,j} = \delta_{i,j} \mu$  and define the form  $\mathbf{s}$  by (8). Then,

$$\langle \mathbf{s}_{(m,\mathbf{e}_0)}, \mathbf{s}_{(n,\mathbf{e}_1)} \rangle = \mathbf{s}(m, n, \mathbf{e}_0, \mathbf{e}_1) = \int t^{m+n} d\mu_{0,1} = 0, \quad m, n \in \mathbb{N}.$$

Hence, the RKHS  $\mathcal{H}$  decomposes naturally as a direct sum  $\mathcal{H}_0 \oplus \mathcal{H}_1$  and the operator  $T$  is accordingly a direct sum of two operators, both having defect indices  $(1, 1)$ , due to the indeterminacy of  $\mu$ . Hence, the defect indices of  $T$  are in this example precisely  $(2, 2)$ .

## 9. Boundedness of the operator $N$ .

**THEOREM 10.** *Suppose one of the equivalent conditions of Theorem 6 holds. Then the operator  $N$  is bounded if and only if for some  $\alpha > 0$  the kernel*

$$\begin{aligned} \mathbf{t}_\alpha(m, n, \mathbf{a}, \mathbf{b}) &\stackrel{\text{def}}{=} a_0 \bar{b}_0 (\alpha s_{0,0}(m, n) - s_{1,1}(m, n)) \\ &+ \alpha a_0 \bar{b}_1 s_{0,1}(m, n) + \alpha a_1 \bar{b}_0 s_{0,1}(m, n) + \alpha a_1 \bar{b}_1 s_{1,1}(m, n) \end{aligned} \quad (33)$$

is a positive definite form on  $\mathbb{N}^2 \times \mathbb{C}^4$ , cf. (3).

**PROOF.** The operator  $N$  is bounded if and only if

$$\langle Nf, Nf \rangle \leq \alpha \langle f, f \rangle, \quad f = \sum_k \mathbf{s}_{(m_k), \mathbf{a}_k} \in \mathcal{D}$$

for some  $\alpha > 0$ . This in turn is equivalent to

$$\alpha \sum_{k,l} \langle \mathbf{s}_{(m_k), \mathbf{a}_k}, \mathbf{s}_{(m_l), \mathbf{a}_l} \rangle - \sum_{k,l} a_{0,k} \bar{a}_{0,l} \langle \mathbf{s}_{(m_k), \mathbf{e}_1}, \mathbf{s}_{(m_l), \mathbf{e}_1} \rangle \geq 0$$

and the left hand side of the above inequality can be written as

$$\sum_{k,l} (\alpha \mathbf{s}(m_l, m_k, \mathbf{a}_l, \mathbf{a}_k) - a_{0,k} \bar{a}_{0,l} \mathbf{s}(m_l, m_k, \mathbf{e}_1, \mathbf{e}_1)) = \sum_{k,l} \mathbf{t}_\alpha(m_l, m_k, \mathbf{a}_l, \mathbf{a}_k).$$

$\square$

*Remark 11.* If

$$s_{0,0}(m, n) = s_{1,1}(m, n) \text{ and } s_{0,1}(m, n) = s_{1,0}(m, n) = 0, \quad m, n = 0, 1, \dots$$

then the form defined in (33) with  $\alpha = 1$  is obviously positive definite. Hence  $N$  is always bounded in this case; confront this with Example 13.

**COROLLARY 12.** *A necessary condition for boundedness of  $N$  is that*

$$\sup_{n \geq 0} \frac{\int t^{2n} d\mu_{11}}{\int t^{2n} d\mu_{00}} < +\infty. \quad (34)$$

**PROOF.** Indeed, note that

$$\|N\mathbf{s}_{(n,\mathbf{a})}\|^2 = |a_0|^2 \|\mathbf{s}_{e_1, n}\|^2 = |a_0|^2 \int t^{2n} d\mu_{11} \quad (35)$$

and

$$\|\mathbf{s}_{(n,\mathbf{a})}\|^2 = |a_0|^2 \int t^{2n} d\mu_{00} + 2\operatorname{Re} \left( a_0 \bar{a}_1 \int t^{2n} d\mu_{01} \right) + |a_1|^2 \int t^{2n} d\mu_{11}. \quad (36)$$

Hence, if  $N$  is bounded then setting  $a_1 = 0$  we get by (35) and (36) that (34) is satisfied.  $\square$

We present an example, showing that  $N$  is not automatically bounded even in the case when the off-diagonal measures  $\mu_{01}$  and  $\mu_{10}$  are zero.

*Example 13.* Let  $\mu_{01}$  and  $\mu_{10}$  be zero measures and let  $d\mu_{11}(t) = t^{2k} d\mu_{00}(t)$  with some  $k \geq 0$ . Then it is a matter of straightforward verification that  $\mathbf{s}_{m,\mathbf{b}} \in \mathcal{D}(N^*)$  for all  $m \in \mathbb{N}$  and  $\mathbf{b} \in \mathbb{C}^2$  and

$$N^* \mathbf{s}_{(m,\mathbf{b})} = b_1 \mathbf{s}_{(e_0, m+2k)}.$$

If now  $\mu_{00}$  is the Gaussian measure with variance one, then

$$\frac{\int t^{2n} d\mu_{11}}{\int t^{2n} d\mu_{11}} = \frac{\int t^{2n+2k} d\mu_{00}}{\int t^{2n} d\mu_{00}} = \frac{(2n+2k-1)!!}{(2n-1)!!} \rightarrow \infty \quad (n \rightarrow \infty).$$

Hence, the necessary condition (34) is violated and consequently  $N$  must not be bounded<sup>12</sup>, although  $\mathcal{D} \subseteq \mathcal{D}(N^*)$ .

The above, when compared with what is in Section 7, shows that boundedness of  $T$  seem to have very little in common with that of  $N$ .

**10. Positive definiteness once more – an open question.** First let us note the following fact.

**PROPOSITION 14.** *If the bisequence  $(s(m, n))_{m,n=0}^\infty$  and the form  $\mathbf{s}$  are in the relation (6) and  $\mathbf{s}$  is a positive definite form, then  $s$  is positive definite, i.e.*

$$\sum_{m,n} \lambda_m \bar{\lambda}_n s(m, n) \geq 0, \quad (\lambda_{m,n})_{m,n} \text{ of finite length.} \quad (37)$$

---

<sup>12</sup> Unbounded nilpotent appear in [8].

PROOF. Note that

$$\begin{aligned}
\sum_{m,n} \lambda_m \bar{\lambda}_n s(m,n) &= \sum_{m,n} \lambda_m \bar{\lambda}_n s(m,n, \mathbf{e}_0, \mathbf{e}_0) \\
&+ \sum_{m,n} \lambda_m \bar{\lambda}_n m s(m-1, n, \mathbf{e}_1, \mathbf{e}_0) \\
&+ \sum_{m,n} \lambda_m \bar{\lambda}_n n s(m, n-1, \mathbf{e}_0, \mathbf{e}_1) \\
&+ \sum_{m,n} \lambda_m \bar{\lambda}_n mn s(m-1, n-1, \mathbf{e}_1, \mathbf{e}_1) \\
&= \sum_{m,n} \mathbf{s}(m, n, \lambda_m \mathbf{e}_0 + (m+1)\lambda_{m+1} \mathbf{e}_1, \lambda_n \mathbf{e}_0) \\
&+ \sum_{m,n} \mathbf{s}(m, n, \lambda_m \mathbf{e}_0 + (m+1)\lambda_{m+1} \mathbf{e}_1, (n+1)\lambda_{n+1} \mathbf{e}_1) \\
&= \sum_{m,n} \mathbf{s}(m, n, \lambda_n \mathbf{e}_0 + (n+1)\lambda_{n+1} \mathbf{e}_1, \lambda_m \mathbf{e}_0 + (m+1)\lambda_{m+1} \mathbf{e}_1) \\
&\geq 0,
\end{aligned}$$

where the last inequality is a consequence of positive definiteness of the form  $\mathbf{s}$ .  $\square$

Now a question appears. Does positive definiteness of  $s$  together with the Helton condition (4) imply existence of *positive definite* form  $\mathbf{s}$  satisfying (6)? In the affirmative case this would give an answer to the Sobolev moment problem purely in terms of the original data  $s$ : a sequence  $s$  is a Sobolev moment sequence if and only if it is positive definite and satisfies (4).

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